Unified Bayesian Decision Theory

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Vocabulary

Let $X = \{a, b, \ldots\}$ be a set partially ordered by the relation $\leq$. Then (if they exist):

1. ab denotes the greatest lower bound on $\{a, b\}$
2. $a \vee b$ denotes the least upper bound on $\{a, b\}$
3. $\neg a$ denotes the (unique) complement of a.
4. $\top$ and $\bot$ respectively denote the greatest and least element in X
5. $a = b$ means $a \leq b$ and $b \leq a$.

A set Y is said to be closed under the operation $\to$ of **conditionalisation** on a set X iff $\forall(a \in X, b \in Y), a \to b \in Y$.

$X_Y =_{def}$ the closure of X under conditionalisation on Y.

Conditional Algebras

The structure $\Psi = \langle Y, \leq, \to \rangle$ is said to be a ***conditional algebra*** based on the structure $\Gamma = \langle X, \leq, \top, \bot \rangle$ iff for all $a, b, c \in X$:

1. Y is closed under conditionalisation on $X \subseteq Y$
2. Γ is a Boolean algebra
3. Ψ is a lattice bounded above and below by $\top$ and $\bot$
4. (C1) $\top \to b = b$
   (C2) $a \to b = a \to ab$
   (M) $(a \to b)(a \to c) = a \to bc$
   (J) $(a \to b) \vee (a \to c) = a \to (b \vee c)$

A conditional algebra Ψ is said to be:

1. **Indicative** iff $a \to (b \to c) = ab \to c$
2. **Normally Bounded** iff
   (i) $a \to \top = \top$
   (ii) $a \to \bot = \bot$ (if $a \neq \bot$)

A **Ramsey algebra** is an indicative, normally bounded conditional algebra.

Factual Prospects: $A = \{f, g, h, \ldots\}$
Conditional prospects: $C = \{\alpha, \beta, \gamma, ..\}$. $C' = C - \{\bot\}$
We assume that $\langle C, \leq, \to \rangle$ forms a Ramsey algebra based on $\langle A, \leq \rangle$.

Bayesian Models

A Bayesian model for a rational agent is a pair of functions, $\langle P, V \rangle$, on C and C' satisfying $\forall(f, g, h, i \in A)$ such that $fg = \bot$:

Belief Axioms

P0 $P(f) \geq 0$

P1 $P(\top) = 1$

P2 $P(f \vee g) = P(f) + P(g)$

P3 $P(f \to h) = P(h|f)$

Desire Axioms

V1 $V(\top) = 0$

V2 $V(f \vee g).P(f \vee g) = V(f).P(f) + V(g).P(g)$

V3 $V(f \to h) = V(h|f).P(f)$

V4 $V((f \to h)(g \to i)) = V(f \to h) + V(g \to i)$

Some Consequences

(i) In the presence of V1 and V2, axioms V3 and V4 are equivalent to the 'state-dependent' SEU hypothesis:

$$V(f \to g)(\neg f \to h) = V(fg).P(f) + V(\neg fh).P(\neg f)$$

(ii) P3 and V3 imply that $\langle P, V \rangle$ are a probability and a desirability over A_f.

(iii) Suppose that $\langle P, V \rangle$ are a probability and a desirability over A_f. Then V3 implies P3.

Value Functions

What characterises Bayesian value functions?

Averaging Slogan: No prospect is better (or worse) than its best (worst) possible realisation.

An real valued function ϕ on a Ramsey algebra is an **Averaging Function** iff
$\phi(fg) \geq \phi(\neg fh) \Rightarrow$

1. $\phi(fg) \geq \phi(fg \vee \neg fh) \geq \phi(f\neg g)$
2. $\phi(fg) \geq \phi((f \to g)(\neg f \to h)) \geq \phi(\neg fh)$

Characterisation Theorem

Suppose:

(A) There exists a real valued functions P_A on A and V on C, that jointly satisfy V3 and V4 and are such that $\forall(\alpha, \beta \in C)$, $V(\alpha) \geq V(\beta) \Leftrightarrow \alpha \succsim \beta$.

(B) The agent's preferences respect:

1. Non-triviality: $\exists(\alpha \in C)$ such that $\alpha \succ \top$
2. Mitigation: $\forall(\alpha, \beta \in C), \exists(f \in A)$ such that $f\alpha \approx \beta$

Then:

If V is an averaging function then there exists a function P on C which agrees with P_A on A and such that $\langle P, V\rangle$ constitutes a Bayesian model of the agent.

Rational Preference

Let $\succsim$ be a (weak preference) relation on C' satisfying $\forall(f, g, h \in A : fg = \bot)$:

1. **Transitivity**: $\alpha \succsim \beta$ and $\beta \succsim \gamma$, then $\alpha \succsim \gamma$

2. **Completeness**: $\alpha \succsim \beta$ or $\beta \succsim \alpha$

3. **Preference for Conditionals**: $f \to \alpha \succsim f \to \beta \Leftrightarrow f\alpha \succsim f\beta$

4. **Independence**:

$$f \to \alpha \succsim f \to \gamma \Leftrightarrow (f \to \alpha)(g \to \beta) \succsim (f \to \gamma)(g \to \beta)$$

5. **Averaging of Disjunctions**: $f \succsim g \Leftrightarrow f \succsim f \vee g \succsim g$

6. **Averaging of Conditionals**:
$f \to g\alpha \succsim f \to \neg g\beta \Leftrightarrow f \to g\alpha \succsim f \to (g \to \alpha)(\neg g \to \beta) \succsim f \to \neg g\beta$

Neutral Prospects

1. ϕ is **neutral** with respect to ψ iff $\phi\psi \approx \psi$.

2. p and q are **equiprobable** iff $\forall\alpha, \beta$ with respect to which they are both neutral:

$$(p \to \alpha)(\neg p \to \beta) \approx (q \to \alpha)(\neg q \to \beta)$$

3. f is **independent** of ϕ iff $\forall\alpha, \beta \in C$:

$$\phi(f \to \alpha)(\neg f \to \beta) \approx (f \to \phi\alpha)(\neg f \to \phi\beta)$$

Let Π be the set of prospects p such that p and $\neg p$ are equiprobable.

Axioms of Neutrality

N1 $\forall(\alpha, \beta, \gamma, \delta \in C, f \in A)$ there exists $p, q \in \Pi$ that are neutral with respect to them and independent of each other and f.

N2 Suppose that $p, q \in \Pi$ are neutral with respect to α and β. Then
$(p \to \alpha)(\neg p \to \beta) \approx (q \to \alpha)(\neg q \to \beta)$

N3 $\forall(\alpha, \beta > \top)$ there exists a partition $\{p_1, p_2, \ldots, p_n\}$ of equiprobable prospects such that for some $i \leq n$, $\alpha > p_i \to \beta$

N4 Suppose that $p \in \Pi$ is neutral with respect to α and β. Then there exists $\gamma, \delta \in C$ such that $(p \to \alpha)(\neg p \to \beta) \approx (p \to \gamma)(\neg p \to \top)$ and $(p \to \alpha)(\neg p \to \delta) \approx \top$.

Defining an Additive Structure

Values

$\boldsymbol{\alpha} =_{def} \{\phi \in C' : \phi \approx \alpha\}$
$\boldsymbol{\alpha} \geq \boldsymbol{\beta} \Leftrightarrow_{def} \forall(\alpha \in \boldsymbol{\alpha}, \beta \in \boldsymbol{\beta}), \alpha \succsim \beta$

Addition

Let p be $\neg p$ be equiprobable and neutral wrt ϕ, α and β. Then

$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} =_{def} \{\phi \in C' : p \to \phi \approx (p \to \alpha)(\neg p \to \beta)\}$$

$$-\boldsymbol{\alpha} =_{def} \{\phi \in C' : (p \to \phi)(\neg p \to \alpha) \approx \bot\}$$

Lemma *Addition*

$\langle \mathbf{C}', \geq, \circ \rangle$ is an (Archimedean) simply ordered group, i.e.

1. $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha}$
2. $(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) \circ \boldsymbol{\gamma} = \boldsymbol{\alpha} \circ (\boldsymbol{\beta} \circ \boldsymbol{\gamma})$
3. $\boldsymbol{\alpha} \circ \top = \boldsymbol{\alpha}$
4. $\boldsymbol{\alpha} \circ (-\boldsymbol{\alpha}) = \top$
5. $\boldsymbol{\alpha} \circ \boldsymbol{\beta} \geq \boldsymbol{\alpha} \circ \boldsymbol{\gamma} \Leftrightarrow \boldsymbol{\beta} \geq \boldsymbol{\gamma}$

Theorem

$\langle \mathbf{C}', \geq, \circ, \top \rangle$ and $\langle \mathbf{C}'_f, \geq, \circ, \mathbf{f} \to \top \rangle$ are isomorphic to $\langle \Re, \geq, +, 0 \rangle$ and if ϕ and ϕ' are any isomorphisms, $\phi = a\phi'$ for some $a > 0$.

(Proof by application of Hölder's theorem).

Representation Theorem

Suppose that $\succsim$ is a complete and transitive relation on C' that respects Independence, Preference for Conditionals and the neutrality axioms. Then:

Existence: there exists a function V on C and a function P on A such that
$\forall(\alpha, b \in C; f, g \in A)$:
(i) $V(\alpha) \geq V(\beta) \Leftrightarrow \alpha \succsim \beta$
(ii) $V(\top) = 0$
(iii) $V(f \to \alpha)(\neg f \to \beta) = V(f \to \alpha) + V(\neg f \to \beta)$
(iv) $V(f \to g) = V(g|f).P(f)$

Uniqueness: if U' and P' are another such a pair of functions satisfying (i) - (iv), then $P' = P$ and $U' = aU$, for some real number $a > 0$.

Corollary: Suppose furthermore that $\succsim$ respects the axioms of averaging. Then $\langle P, V \rangle$ constitute a Bayesian model.